Complexity of two-variable Dependence Logic and IF-Logic*

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January 19, 2013

We study the two-variable fragments D^2 and IF^2 of dependence logic and independence-friendly logic. We consider the satisfiability and finite satisfiability problems of these logics and show that for D^2, both problems are *NEXPTIME*-complete, whereas for IF^2, the problems are Π^0_1 and Σ^0_1-complete, respectively. We also show that D^2 is strictly less expressive than IF^2 and that already in D^2, equicardinality of two unary predicates and infinity can be expressed (the latter in the presence of a constant symbol).

This is an extended version of a publication in the proceedings of the 26th Annual IEEE Symposium on Logic in Computer Science (LICS 2011).

Keywords: dependence logic, independence-friendly logic, two-variable logic, decidability, complexity, satisfiability, expressivity

ACM Subject Classifiers: F.4.1 Computability theory, Model theory; F.1.3 Reducibility and completeness

1 Introduction

The satisfiability problem of first-order logic FO was shown to be undecidable in [Chu36, Tur36], and ever since, logicians have been searching for decidable fragments of FO. Henkin [Hen67] was the first to consider the logics FO^k, i.e., the fragments of first-order logic with k variables. The fragments FO^k, for $k \geq 3$, were easily seen to be undecidable but the case for $k = 2$ remained open. Scott [Sco62] then showed that FO^2 without equality is decidable. Mortimer [Mor75] extended the result to FO^2 with equality and showed that every satisfiable FO^2 formula has a model whose size is doubly exponential in the length of the formula. His result established that the satisfiability

*This work was supported by grants 127661, 129208, 129761, 129892 and 138163 of the Academy of Finland and by DAAD grant 50740539

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Logic	*Complexity of* SAT / FINSAT	*References*
FO, FO^3	Π^0_1 / Σ^0_1	[Chu36, Tur36]
ESO, D, IF	Π^0_1 / Σ^0_1	Remark 2.1, [Chu36, Tur36]
FO^2	*NEXPTIME*	[GKV97]
FOC^2	*NEXPTIME*	[PH05]
D^2	*NEXPTIME*	Theorem 5.2
$\mathsf{FO}^2(\mathrm{I})$	Σ^1_1-hard / Σ^0_1	[GOR97b]
IF^2	Π^0_1 / Σ^0_1	Theorems 4.13, 4.20

Table 1: Complexity of satisfiability for various logics.
The results are completeness results for the full relational vocabulary.

and finite satisfiability problems of FO^2 are contained in *2NEXPTIME*. Finally, Grädel, Kolaitis and Vardi [GKV97] improved the result of Mortimer by establishing that every satisfiable FO^2 formula has a model of exponential size. Furthermore, they showed that the satisfiability problem for FO^2 is *NEXPTIME*-complete.

The decidability of the satisfiability problem of various extensions of FO^2 has been studied (e.g. [GOR97b, GO99, EVW02, KO05]). One such interesting extension FOC^2 is acquired by extending FO^2 with counting quantifiers $\exists^{\geq i}$. The meaning of a formula of the form $\exists^{\geq i}x\phi(x)$ is that $\phi(x)$ is satisfied by at least i distinct elements. The satisfiability problem for the logic FOC^2 was shown to be decidable by Grädel et al. [GOR97a], and shown to be in *2NEXPTIME* by Pacholski et al. [PST97]. Finally, Pratt-Hartmann [PH05] established that the problem is *NEXPTIME*-complete. We will later use the result of Pratt-Hartmann to determine the complexity of the satisfiability problem of the two-variable fragment of dependence logic.

In this article we study the satisfiability of the two-variable fragments of independence-friendly logic (IF) and dependence logic (D). The logics IF and D are conservative extensions of FO, i.e., they agree with FO on sentences which syntactically are FO-sentences. We thereby contribute to the understanding of the satisfiability problems of extensions of FO^2. We briefly recall the history of IF and D. In first-order logic the order in which quantifiers are written determines dependence relations between variables. For example, when using game theoretic semantics to evaluate the formula

$$\forall x_0 \exists x_1 \forall x_2 \exists x_3\, \phi,$$

the choice for x_1 depends on the value for x_0, and the choice for x_3 depends on the value of both universally quantified variables x_0 and x_2. The characteristic feature of D and IF is that in these logics it is possible to express dependencies between variables that cannot be expressed in FO. The first step in this direction was taken by Henkin [Hen61] with his partially ordered quantifiers

$$\begin{pmatrix} \forall x_0 & \exists x_1 \\ \forall x_2 & \exists x_3 \end{pmatrix} \phi, \tag{1}$$

where x_1 depends only on x_0 and x_3 depends only on x_2. Enderton [End70] and Walkoe [Wal70] observed that exactly the properties definable in existential second-order logic (ESO) can be expressed with partially ordered quantifiers. The second step was taken by Hintikka and Sandu [HS89, Hin96], who introduced independence-friendly logic, which extends FO in terms of so-called slashed quantifiers. For example, in

$$\forall x_0 \exists x_1 \forall x_2 \exists x_3 / \forall x_0 \phi,$$

the quantifier $\exists x_3 / \forall x_0$ means that x_3 is "independent" of x_0 in the sense that a choice for the value of x_3 should not depend on what the value of x_0 is. The semantics of IF was first formulated in game theoretic terms, and IF can be regarded as a game theoretically motivated generalization of FO. Whereas the semantic game for FO is a game of perfect information, the game for IF is a game of imperfect information. The so-called team semantics of IF, also used in this paper, was introduced by Hodges [Hod97a].

Dependence logic, introduced by Väänänen [Vää07], was inspired by IF-logic, but the approach of Väänänen provided a fresh perspective on quantifier dependence. In dependence logic the dependence relations between variables are written in terms of novel atomic dependence formulas. For example, the partially ordered quantifier (1) can be expressed in dependence logic as follows

$$\forall x_0 \exists x_1 \forall x_2 \exists x_3 (=(x_2, x_3) \wedge \phi).$$

The atomic formula $=(x_2, x_3)$ has the explicit meaning that x_3 is completely determined by x_2 and nothing else.

In recent years, research related to IF and D has been active. A variety of closely related logics have been defined and various applications suggested, see e.g. [Abr07, BK05, GV10, LV10, Sev09, VH10]. While both IF and D are known to be equi-expressive to ESO, the relative strengths and weaknesses of the two different logics in relation to applications is not understood well. In this article we take a step towards a better understanding of this matter. After recalling some basic properties in Section 2, we compare the expressivity of the finite variable fragments of D and IF in Section 3. We show that there is an effective translation from D^2 to IF^2 (Theorem 3.1) and from IF^2 to D^3 (Theorem 3.2). We also show that IF^2 is strictly more expressive than D^2 (Proposition 3.5). This result is a by-product of our proof in Section 4 that the satisfiability problem of IF^2 is undecidable (Theorem 4.13 shows Π^0_1-completeness). The proof can be adapted to the context of finite satisfiability, i.e., the problem of determining for a given formula ϕ whether there is a finite structure $\mathfrak{A}$ such that $\mathfrak{A} \models \phi$ (Theorem 4.20 shows Σ^0_1-completeness). The undecidability proofs are based on tiling arguments. Finally, in Section 5, we study the decidability of the satisfiability and finite satisfiability problems of D^2. For this purpose we reduce the problems to the (finite) satisfiability problem for FOC^2 (Theorem 5.1) and thereby show that they are *NEXPTIME*-complete (Theorem 5.2). Table 1 gives an overview of previously-known as well as new complexity results.

2 Preliminaries

In this section we recall the basic concepts and results relevant for this article.

The domain of a structure $\mathfrak{A}$ is denoted by A. We assume that the reader is familiar with first-order logic FO. The extension of FO in terms of counting quantifiers $\exists^{\geq i}$ is denoted by FOC. We also consider the extension $\mathsf{FO}(\mathrm{I})$ of FO by the Härtig quantifier I. The interpretation of the quantifier I is defined by the clause

$$\mathfrak{A}, s \models \mathrm{I}\, xy(\phi(x), \psi(y)) \Leftrightarrow |\phi(x)^{\mathfrak{A},s}| = |\psi(y)^{\mathfrak{A},s}|,$$

where $\phi(x)^{\mathfrak{A},s} := \{a \in A \mid \mathfrak{A}, s \models \phi(a)\}$. The k-variable fragments FO^k, FOC^k, and $\mathsf{FO}^k(\mathrm{I})$ are the fragments of FO, FOC, and $\mathsf{FO}(\mathrm{I})$ with formulas in which at most k, say $x_1, \ldots, x_k$, distinct variables appear. In the case $k = 2$, we denote these variables by x and y. The existential fragment of second-order logic is denoted by ESO. For logics $\mathcal{L}$ and $\mathcal{L}'$, we write $\mathcal{L} \leq \mathcal{L}'$ if for every sentence ϕ of $\mathcal{L}$ there is a sentence ϕ^* of $\mathcal{L}'$ such that for all structures $\mathfrak{A}$ it holds that $\mathfrak{A} \models \phi$ iff $\mathfrak{A} \models \phi^*$. We write $\mathcal{L}\equiv\mathcal{L}'$ if $\mathcal{L} \leq \mathcal{L}'$ and $\mathcal{L}' \leq \mathcal{L}$.

We assume that the reader is familiar with the basics of computational complexity theory. In this article we are interested in the complexity of the satisfiability problems of various logics. For any logic $\mathcal{L}$ the *satisfiability problem* $\textsc{Sat}(\mathcal{L})$ is defined as

$$\textsc{Sat}(\mathcal{L}) := \{\phi \in \mathcal{L} \mid \text{there is a structure } \mathfrak{A} \text{ such that } \mathfrak{A} \models \phi\}.$$

The finite satisfiability problem $\textsc{FinSat}(\mathcal{L})$ is the analogue of $\textsc{Sat}(\mathcal{L})$ in which we require the structure $\mathfrak{A}$ to be finite. The following observation will be useful later.

Remark 2.1. *If ϕ is a formula over the vocabulary τ and*

$$\psi := \exists R_1 \ldots \exists R_n \exists f_1 \ldots \exists f_m \phi$$

with $R_1, \ldots, R_n, f_1, \ldots, f_m \in \tau$, then ϕ is satisfiable iff the second-order formula ψ is satisfiable.

2.1 The logics D and IF

In this section we define independence-friendly logic and dependence logic and recall some related basic results. For IF we follow the exposition of [CDJ09] and the forthcoming monograph [MSS11].

Definition 2.2. The syntax of IF extends the syntax of FO defined in terms of $\vee$, $\wedge$, $\neg$, $\exists$ and $\forall$, by adding quantifiers of the form

$$\exists x/W \phi$$
$$\forall x/W \phi$$

called slashed quantifiers, where x is a first-order variable, W a finite set of first-order variables and ϕ a formula.

Definition 2.3 ([Vää07])**.** The syntax of D extends the syntax of FO, defined in terms of $\vee$, $\wedge$, $\neg$, $\exists$ and $\forall$, by new atomic (dependence) formulas of the form

$$=(t_1, \dots, t_n),$$

where $t_1, \dots, t_n$ are terms.

The set $\mathrm{Fr}(\phi)$ of free variables of a formula $\phi \in \mathsf{D} \cup \mathsf{IF}$ is defined as for first-order logic except that we have the new cases

$$\begin{aligned}
&\mathrm{Fr}(=(t_1, \dots, t_n)) = \mathrm{Var}(t_1) \cup \dots \cup \mathrm{Var}(t_n) \\
&\mathrm{Fr}(\exists x/W\psi) = W \cup (\mathrm{Fr}(\psi) \setminus \{x\}) \\
&\mathrm{Fr}(\forall x/W\psi) = W \cup (\mathrm{Fr}(\psi) \setminus \{x\})
\end{aligned}$$

where $\mathrm{Var}(t_i)$ is the set of variables occurring in the term t_i. If $\mathrm{Fr}(\phi) = \emptyset$, we call ϕ a sentence.

Definition 2.4. Let τ be a relational vocabulary, i.e., τ does not contain function or constant symbols.

a) The *two-variable independence-friendly* logic $\mathsf{IF}^2(\tau)$ is generated from τ according to the following grammar:

$$\begin{aligned}
\phi ::= &t_1 = t_2 \mid R(t_1, \dots, t_n) \mid \neg t_1 = t_2 \mid \neg R(t_1, \dots, t_n) \mid \\
&(\phi \wedge \phi) \mid (\phi \vee \phi) \mid \forall x\phi \mid \forall y\phi \mid \exists x/W\phi \mid \exists y/W\phi
\end{aligned}$$

b) The *two-variable dependence* logic $\mathsf{D}^2(\tau)$ is generated from τ according to the following grammar:

$$\begin{aligned}
\phi ::= &t_1 = t_2 \mid R(t_1, \dots, t_n) \mid \neg t_1 = t_2 \mid \neg R(t_1, \dots, t_n) \mid =(t_1, t_2) \mid \neg=(t_1, t_2) \mid \\
&=(t_1) \mid \neg=(t_1) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \forall x\phi \mid \forall y\phi \mid \exists x\phi \mid \exists y\phi
\end{aligned}$$

Here $R \in \tau$ is an n-ary relation symbol, $W \subseteq \{x, y\}$ and $t_1, \dots, t_n \in \{x, y\}$. We identify existential first-order quantifiers with existential quantifiers with empty slash sets, and therefore if $W = \emptyset$ we simply write $\exists x\phi(x)$ instead of $\exists x/W\phi(x)$. When τ is clear we often leave it out. To simplify notation, we assume in the following that the relation symbols $R \in \tau$ are at most binary.

Note that in Definition 2.4 we have only defined formulas in negation normal form and for that reason we do not need the slashed universal quantifier in IF^2 [Hod97a]. Defining syntax in negation normal form is customary in IF and D. A formula ϕ with arbitrary negations is considered an abbreviation of the negation normal form formula ψ obtained from ϕ by pushing the negations to the atomic level in the same fashion as in first-order logic. It is important to note that the game theoretically motivated negation $\neg$ of D and IF does not satisfy the law of excluded middle and is therefore not the classical Boolean negation. This is manifested by the existence of sentences ϕ such that for some $\mathfrak{A}$ we have $\mathfrak{A} \not\models \phi$ and $\mathfrak{A} \not\models \neg\phi$.

In order to define the semantics of IF and D, we first need to define the concept of a *team*. Let $\mathfrak{A}$ be a model with the domain A. *Assignments* over $\mathfrak{A}$ are finite functions that map variables to elements of A. The value of a term t in an assignment s is denoted by $t^{\mathfrak{A}}\langle s\rangle$. If s is an assignment, x a variable, and $a \in A$, then $s(a/x)$ denotes the assignment (with the domain $\mathrm{dom}(s) \cup \{x\}$) which agrees with s everywhere except that it maps x to a.

Let A be a set and $\{x_1, \dots, x_k\}$ a finite (possibly empty) set of variables. A *team* X of A with the domain $\mathrm{dom}(X) = \{x_1, \dots, x_k\}$ is any set of assignments from the variables $\{x_1, \dots, x_k\}$ into the set A. We denote by $\mathrm{rel}(X)$ the k-ary relation of A corresponding to X

$$\mathrm{rel}(X) = \{(s(x_1), \dots, s(x_k)) \mid s \in X\}.$$

If X is a team of A, and $F\colon X \to A$, we use $X(F/x)$ to denote the team $\{s(F(s)/x) \mid s \in X\}$ and $X(A/x)$ the team $\{s(a/x) \mid s \in X \text{ and } a \in A\}$. For a set $W \subseteq \mathrm{dom}(X)$ we call F *W-independent* if for all $s, s' \in X$ with $s(x) = s'(x)$ for all $x \in \mathrm{dom}(X) \setminus W$ we have that $F(s) = F(s')$.

We are now ready to define the semantics of IF and D.

Definition 2.5 ([Hod97a, Vää07])**.** Let $\mathfrak{A}$ be a model and X a team of A. The satisfaction relation $\mathfrak{A} \models_X \phi$ is defined as follows:

1. If ϕ is a first-order literal, then $\mathfrak{A} \models_X \phi$ iff for all $s \in X$: $\mathfrak{A}, s \models_{\mathsf{FO}} \phi$.
2. $\mathfrak{A} \models_X \psi \wedge \phi$ iff $\mathfrak{A} \models_X \psi$ and $\mathfrak{A} \models_X \phi$.
3. $\mathfrak{A} \models_X \psi \vee \phi$ iff there exist teams Y and Z such that $X = Y \cup Z$, $\mathfrak{A} \models_Y \psi$ and $\mathfrak{A} \models_Z \phi$.
4. $\mathfrak{A} \models_X \exists x\psi$ iff $\mathfrak{A} \models_{X(F/x)} \psi$ for some $F\colon X \to A$.
5. $\mathfrak{A} \models_X \forall x\psi$ iff $\mathfrak{A} \models_{X(A/x)} \psi$.

For IF we further have the following rules:

6. $\mathfrak{A} \models_X \exists x/W\phi$ iff $\mathfrak{A} \models_{X(F/x)} \phi$ for some W-independent function $F : X \to A$.
7. $\mathfrak{A} \models_X \forall x/W\phi$ iff $\mathfrak{A} \models_{X(A/x)} \phi$.

And for D we have the additional rules:

8. $\mathfrak{A} \models_X {=}(t_1, \dots, t_n)$ iff for all $s, s' \in X$ such that $t_1^{\mathfrak{A}}\langle s\rangle = t_1^{\mathfrak{A}}\langle s'\rangle, \dots, t_{n-1}^{\mathfrak{A}}\langle s\rangle = t_{n-1}^{\mathfrak{A}}\langle s'\rangle$, we have $t_n^{\mathfrak{A}}\langle s\rangle = t_n^{\mathfrak{A}}\langle s'\rangle$.
9. $\mathfrak{A} \models_X \neg{=}(t_1, \dots, t_n)$ iff $X = \emptyset$.

Above, we assume that the domain of X contains $\mathrm{Fr}(\phi)$. Finally, a sentence ϕ is true in a model $\mathfrak{A}$ ($\mathfrak{A} \models \phi$) if $\mathfrak{A} \models_{\{\emptyset\}} \phi$.

From Definition 2.5 it follows that many familiar propositional equivalences of connectives do not hold in D and IF. For example, the idempotence of disjunction fails, which can be used to show that the distributivity laws of disjunction and conjunction do not hold either. We refer to [Vää07, Section 3.3] for a detailed exposition on propositional equivalences of connectives in D (and also IF). Another feature of Definition 2.5 is that $\mathfrak{A} \models_{\emptyset} \phi$ for all $\mathfrak{A}$ and all formulas ϕ of D and IF. This observation is important in noting that, for sentences ϕ and ψ, the interpretation of $\phi \vee \psi$ coincides with the classical disjunction of ϕ and ψ.

2.2 Basic properties of D and IF

In this section we recall some basic properties of D and IF.

Let X be a team with the domain $\{x_1, \ldots, x_k\}$ and $V \subseteq \{x_1, \ldots, x_k\}$. We denote by $X \upharpoonright V$ the team $\{s \upharpoonright V \mid s \in X\}$ with the domain V. The following proposition shows that the truth of a D-formula depends only on the interpretations of the variables occurring free in the formula.

Proposition 2.6 ([Vää07, CDJ09]). *Let $\phi \in \mathsf{D}$ be any formula or $\phi \in \mathsf{IF}$ a sentence. If $V \supseteq \mathrm{Fr}(\phi)$, then $\mathfrak{A} \models_X \phi$ if and only if $\mathfrak{A} \models_{X \upharpoonright V} \phi$.*

The analogue of Proposition 2.6 does not hold for open formulas of IF. In other words, the truth of an IF-formula may depend on the interpretations of variables that do not occur in the formula. For example, the truth of the formula ϕ

$$\phi = \exists x/\{y\}(x = y)$$

in a team X with domain $\{x, y, z\}$ depends on the values of z in X, although z does not occur in ϕ.

The following fact is a fundamental property of all formulas of D and IF:

Proposition 2.7 ([Vää07, Hod97a], Downward closure). *Let ϕ be a formula of D or IF, $\mathfrak{A}$ a model, and $Y \subseteq X$ teams. Then $\mathfrak{A} \models_X \phi$ implies $\mathfrak{A} \models_Y \phi$.*

The expressive power of sentences of D and IF coincides with that of existential second-order sentences:

Theorem 2.8. $\mathsf{D} \equiv \mathsf{IF} \equiv \mathsf{ESO}$.

Proof. The fact $\mathsf{ESO} \leq \mathsf{D}$ (and $\mathsf{ESO} \leq \mathsf{IF}$) is based on the analogous result of [End70, Wal70] for partially ordered quantifiers. For the converse inclusions, see [Vää07] and [Hod97b]. □

Proposition 2.9 ([Vää07, Hod97a]). *Let ϕ be a formula of D or IF without dependence atoms and without slashed quantifiers, i.e., ϕ is syntactically a first-order formula. Then for all $\mathfrak{A}$, X and s:*

1. *$\mathfrak{A} \models_{\{s\}} \phi$ iff $\mathfrak{A}, s \models_{\mathsf{FO}} \phi$.*
2. *$\mathfrak{A} \models_X \phi$ iff for all $s \in X : \mathfrak{A}, s \models_{\mathsf{FO}} \phi$.*

3 Comparison of IF and D

In this section we show that

$$\mathsf{D}^2 < \mathsf{IF}^2 \leq \mathsf{D}^3.$$

We also further discuss the expressive powers and other logical properties of D^2 and IF^2.

Lemma 3.1. *For any formula $\phi \in \mathsf{D}^2$ there is a formula $\phi^* \in \mathsf{IF}^2$ such that for all structures $\mathfrak{A}$ and teams X, where* $\mathrm{dom}(X) = \{x, y\}$, *it holds that*

$$\mathfrak{A} \models_X \phi \quad \Leftrightarrow \quad \mathfrak{A} \models_X \phi^*.$$

Proof. The translation $\phi \mapsto \phi^*$ is defined as follows. For first-order literals the translation is the identity, and negations of dependence atoms are translated by $\neg x = x$. The remaining cases are defined as follows:

$$\begin{array}{rcl} =\!(x) & \mapsto & \exists y/\{x,y\}(x = y) \\ =\!(x,y) & \mapsto & \exists x/\{y\}(x = y) \\ \phi \wedge \psi & \mapsto & \phi^* \wedge \psi^* \\ \phi \vee \psi & \mapsto & \phi^* \vee \psi^* \\ \exists x\phi & \mapsto & \exists x\phi^* \\ \forall x\phi & \mapsto & \forall x\phi^* \end{array}$$

The claim of the lemma can now be proved using induction on ϕ. The only non-trivial cases are the dependence atoms. We consider the case where ϕ is of the form $=\!(x, y)$.

Let us assume that $\mathfrak{A} \models_X \phi$. Then there is a function $F\colon A \to A$ such that

$$\text{for all } s \in X: \; s(y) = F(s(x)). \tag{2}$$

Define now $F'\colon X \to A$ as follows:

$$F'(s) := F(s(x)). \tag{3}$$

F' is $\{y\}$-independent since, if $s(x) = s'(x)$, then

$$F'(s) = F(s(x)) = F(s'(x)) = F'(s').$$

It remains to show that

$$\mathfrak{A} \models_{X(F'/x)} (x = y). \tag{4}$$

Let $s \in X(F'/x)$. Then

$$s = s'(F'(s')/x) \text{ for some } s' \in X. \tag{5}$$

Now

$$s(x) \overset{(5)}{=} F'(s') \overset{(3)}{=} F(s'(x)) \overset{(2)}{=} s'(y) \overset{(5)}{=} s(y).$$

Therefore, (4) holds, and hence also

$$\mathfrak{A} \models_X \exists x/\{y\}(x = y).$$

Suppose then that $\mathfrak{A} \not\models_X \phi$. Then there must be $s, s' \in X$ such that $s(x) = s'(x)$ and $s(y) \neq s'(y)$. We claim now that

$$\mathfrak{A} \not\models_X \exists x/\{y\}(x = y). \tag{6}$$

Let $F\colon X \to A$ be an arbitrary $\{y\}$-independent function. Then, by $\{y\}$-independence, $F(s) = F(s')$ and since additionally $s(y) \neq s'(y)$, we have

$$s(F(s)/x)(x) = F(s) \neq s(y) = s(F(s)/x)(y)$$

or

$$s'(F(s')/x)(x) = F(s') \neq s'(y) = s'(F(s')/x)(y).$$

This implies that

$$\mathfrak{A} \not\models_{X(F/x)} (x = y),$$

since $s(F(s)/x)$, $s'(F(s')/x) \in X(F/x)$.

Since F was arbitrary, we may conclude that (6) holds. $\square$

Next we show a translation from IF^2 to D^3.

Lemma 3.2. *For any formula $\phi \in \mathsf{IF}^2$ there is a formula $\phi^* \in \mathsf{D}^3$ such that for all structures $\mathfrak{A}$ and teams X, where $\mathrm{dom}(X) = \{x, y\}$, it holds that*

$$\mathfrak{A} \models_X \phi \quad\Leftrightarrow\quad \mathfrak{A} \models_X \phi^*.$$

Proof. The claim follows by the following translation $\phi \mapsto \phi^*$: For atomic and negated atomic formulas the translation is the identity, and for propositional connectives and first-order quantifiers it is defined in the obvious inductive way. The only non-trivial cases are the slashed quantifiers:

$$\begin{aligned}
\exists x/\{y\}\psi &\;\mapsto\; \exists z(x = z \wedge \exists x({=}(z, x) \wedge \psi^*)),\\
\exists x/\{x\}\psi &\;\mapsto\; \exists x({=}(y, x) \wedge \psi^*),\\
\exists x/\{x, y\}\psi &\;\mapsto\; \exists x({=}(x) \wedge \psi^*).
\end{aligned}$$

Again, the claim can be proved using induction on ϕ. We consider the case where ϕ is of the form $\exists x/\{y\}\psi$. Assume $\mathfrak{A} \models_X \phi$. Then there is a $\{y\}$-independent function $F\colon X \to A$ such that

$$\mathfrak{A} \models_{X(F/x)} \psi. \tag{7}$$

By $\{y\}$-independence, $s(x) = s'(x)$ implies that $F(s) = F(s')$ for all $s, s' \in X$. Our goal is to show that

$$\mathfrak{A} \models_X \exists z(x = z \wedge \exists x({=}(z, x) \wedge \psi^*)). \tag{8}$$

Now, (8) holds if for $G\colon X \to A$ defined by $G(s) = s(x)$ for all $s \in X$ it holds that

$$\mathfrak{A} \models_{X(G/z)} \exists x(=(z,x) \wedge \psi^*). \tag{9}$$

Define $F'\colon X(G/z) \to A$ by $F'(s) = F(s \restriction \{x,y\})$. Now we claim that

$$\mathfrak{A} \models_{X(G/z)(F'/x)} =(z,x) \wedge \psi^*,$$

implying (9) and hence (8).

First we show that

$$\mathfrak{A} \models_{X(G/z)(F'/x)} =(z,x). \tag{10}$$

At this point it is helpful to note that every $s \in X(G/z)(F'/x)$ arises from an $s' \in X$ by first copying the value of x to z and then replacing the value of x by $F(s \restriction \{x,y\})$, i.e., that $s(z) = s'(G(s')/z)(z) = G(s') = s'(x)$ and $s(x) = F(s')$. Now, to show (10), let $s_1, s_2 \in X(G/z)(F'/x)$ with $s_1(z) = s_2(z)$ and let $s_1', s_2' \in X$ as above, i.e., s_1 (resp. s_2) arises from s_1' (resp. s_2'). Then it follows that $s_1'(x) = s_2'(x)$. Hence, by $\{y\}$-independence, $F(s_1') = F(s_2')$, implying that $s_1(x) = F(s_1') = F(s_2') = s_2(x)$ which proves (10). Let us then show that

$$\mathfrak{A} \models_{X(G/z)(F'/x)} \psi^*. \tag{11}$$

Note first that by the definition of the mapping $\phi \mapsto \phi^*$ the variable z cannot appear free in ψ^*. By Proposition 2.6, the satisfaction of any D-formula θ only depends on those variables in a team that appear free in θ, therefore (11) holds iff

$$\mathfrak{A} \models_{X(G/z)(F'/x)\restriction\{x,y\}} \psi^*. \tag{12}$$

We have chosen G and F' in such a way that

$$X(G/z)(F'/x) \restriction \{x,y\} = X(F/x),$$

hence (12) now follows from (7) and the induction hypothesis.

We omit the proof of the converse implication which is analogous. $\square$

For sentences, Lemmas 3.1 and 3.2 now imply the following.

Theorem 3.3. $\mathsf{D}^2 \leq \mathsf{IF}^2 \leq \mathsf{D}^3$

Proof. The claim follows by Lemmas 3.1 and 3.2. First of all, if ϕ is a sentence of IF or D, then, by Proposition 2.6, for every model $\mathfrak{A}$ and team $X \neq \emptyset$

$$\mathfrak{A} \models_X \phi \text{ iff } \mathfrak{A} \models_{\{\emptyset\}} \phi. \tag{13}$$

It is important to note that, even if $\phi \in \mathsf{D}^2$ is a sentence, it may happen that ϕ^* has free variables since variables in W are regarded as free in subformulas of ϕ^* of the form $\exists x/W\psi$. However, this is not a problem. Let Y be the set of all assigments of $\mathfrak{A}$ with the domain $\{x,y\}$. Now

$$\begin{aligned} \mathfrak{A} \models_{\{\emptyset\}} \phi \quad &\text{iff} \quad \mathfrak{A} \models_Y \phi \text{ iff } \mathfrak{A} \models_Y \forall x \forall y \phi \\ &\text{iff} \quad \mathfrak{A} \models_Y \forall x \forall y \phi^* \text{ iff } \mathfrak{A} \models_{\{\emptyset\}} \forall x \forall y \phi^*, \end{aligned}$$

where the first and the last equivalence hold by (13), the second by the semantics of the universal quantifier and the third by Lemma 3.1. An analogous argument can be used to show that for every sentence $\phi \in \mathsf{IF}^2$ there is an equivalent sentence of the logic D^3. $\square$

3.1 Examples of properties definable in D^2

We end this section with examples of definable classes of structures in D^2 (and in IF^2 by Theorem 3.3).

Proposition 3.4. *The following properties can be expressed in D^2:*

a) For unary relation symbols P and Q, D^2 can express $|P| = |Q|$. This shows $\mathsf{D}^2 \not\leq \mathsf{FO}$.

b) If the vocabulary of $\mathfrak{A}$ contains a constant c, then D^2 can express that A is infinite.

c) $|A| \leq k$ can be expressed already in D^1.

Proof. Let us first consider part a). Clearly, it suffices to express $|P| \leq |Q|$. Define ϕ by

$$\phi := \forall x \exists y (=(y, x) \wedge (\neg P(x) \vee Q(y))).$$

Now, $\mathfrak{A} \models \phi$ iff there is an injective function $F \colon A \to A$ such that $F[P^{\mathfrak{A}}] \subseteq Q^{\mathfrak{A}}$ iff $|P^{\mathfrak{A}}| \leq |Q^{\mathfrak{A}}|$.

For part b), we use the same idea as above. Define ψ by

$$\psi := \forall x \exists y (=(y, x) \wedge \neg c = y).$$

Now, $\mathfrak{A} \models \psi$ iff there is an injective function $F \colon A \to A$ such that $c^{\mathfrak{A}} \notin F[A]$ iff A is infinite.

Finally, we show how to express the property from part c). Define θ as

$$\forall x (\bigvee_{1 \leq i \leq k} \chi_i),$$

where χ_i is $=(x)$. It is now immediate that $\mathfrak{A} \models \theta$ iff $|A| \leq k$. □

It is interesting to note that, although part a) holds, the difference in SAT-complexity of $\mathsf{FO}^2(\mathrm{I})$ and D^2 is a major one. The former is Σ^1_1-hard [GOR97b] whereas the latter is decidable – as is shown in section 5. Part a) also implies that D^2 does not have a zero-one law, since the property $|P| \leq |Q|$ (which can be expressed in D^2) has the limit probability $\frac{1}{2}$.

Proposition 3.5. $\mathsf{D}^2 < \mathsf{IF}^2$. *This holds already in the finite.*

Proof. The property of being grid-like (see Definition 4.9) can be expressed in IF^2 but not in D^2 since D^2 is decidable by Theorem 5.2. In the finite, there exists no D^2 sentence equivalent to the IF^2 sentence ϕ_{torus} (see Section 4.1), since the finite satisfiability problem of D^2 is decidable. □

4 Satisfiability for IF^2 is undecidable

In this section we will use tiling problems, introduced by Hao Wang in [Wan61], to show the undecidability of $\textsc{Sat}(\mathsf{IF}^2)$ as well as $\textsc{FinSat}(\mathsf{IF}^2)$.

In this paper a *Wang tile* is a square in which each edge is assigned a color. It is a square that has four colors (up, right, down, left). We say that a set of tiles can tile the $\mathbb{N} \times \mathbb{N}$ plane if a tile can be placed on every point $(i, j) \in \mathbb{N} \times \mathbb{N}$ s.t. the right color of the tile in (i, j) is the same as the left color of the tile in $(i + 1, j)$ and the up color of the tile in (i, j) is the same as the down color in the tile in $(i, j+1)$. Notice that turning and flipping tiles is not allowed.

We then define some specific structures needed later.

Definition 4.1. The model $\mathfrak{G} := (G, V, H)$ where

- $G = \mathbb{N} \times \mathbb{N}$,
- $V = \{((i, j), (i, j + 1)) \subseteq G \times G \mid i, j \in \mathbb{N}\}$ and
- $H = \{((i, j), (i + 1, j)) \subseteq G \times G \mid i, j \in \mathbb{N}\}$

is called the *grid*.

A finite model $\mathfrak{D} = (D, V, H, V', H')$ where

- $D = \{0, \ldots, n\} \times \{0, \ldots, m\}$,
- $V = \{((i, j), (i, j + 1)) \subseteq D \times D \mid i \leq n, j < m\}\}$,
- $H = \{((i, j), (i + 1, j)) \subseteq D \times D \mid i < n, j \leq m\}$,
- $V' = \{((i, m), (i, 0)) \subseteq D \times D \mid i \leq n\}$ and
- $H' = \{((n, j), (0, j)) \subseteq D \times D \mid j \leq m\}$

is called a *torus*.

Definition 4.2. A set of *colors* C is defined to be an arbitrary finite subset of the natural numbers. The set of all *(Wang) tiles* over C is C^4, i.e., a tile is an ordered list of four colors, interpreted as the colors of the four edges of the tile in the order top, right, bottom and left.

Let C be a set of colors, $T \subseteq C^4$ a finite set of tiles and $\mathfrak{A} = (A, V, H)$ a first-order structure with binary relations V and H interpreted as vertical and horizontal successor relations. Then a T-*tiling* of $\mathfrak{A}$ is a total function $t\colon A \to T$ such that for all $x, y \in A$ it holds that

i) $(t(x))_0 = (t(y))_2$ if $(x, y) \in V$, i.e., the top color of x matches the bottom color of y, and

ii) $(t(x))_1 = (t(y))_3$ if $(x, y) \in H$, i.e., the right color of x matches the left color of y.

Next we define the tiling problem for a structure $\mathfrak{A} = (A, V, H)$.

Definition 4.3 (Tiling)**.** A structure $\mathfrak{A} = (A, V, H)$ is called *T-tilable* iff there is a T-tiling of $\mathfrak{A}$.

For any structure $\mathfrak{A} = (A, V, H)$ we define the problem

$$\textsc{Tiling}(\mathfrak{A}) := \{T \mid \mathfrak{A} \text{ is } T\text{-tilable}\}.$$

We say that a structure $\mathfrak{B} = (B, V, H, V', H')$ is T-tilable if and only if the structure $(B, V \cup V', H \cup H')$ is T-tilable. Hence a torus $\mathfrak{D} = (D, V, H, V', H')$ is T-tilable if and only if the structure $(D, V \cup V', H \cup H')$ is T-tilable. Now we define the problem

$$\textsc{Tiling}(\text{Torus}) := \{T \mid \text{there is a torus } \mathfrak{D} \text{ that is } T\text{-tilable}\}.$$

Note that the set $\textsc{Tiling}(\mathfrak{G})$ consists of all T such that there is a T-tiling of the infinite grid and $\textsc{Tiling}(\text{Torus})$ consists of all T such that there is a *periodic* T-tiling of the grid. Further note that $\textsc{Tiling}(\text{Torus})$ cannot be expressed in the form $\textsc{Tiling}(\mathfrak{D})$ for a fixed torus $\mathfrak{D}$ since a fixed torus has a fixed size and we want the problem to be the question whether there is a torus of *any* size.

We will later use the following two theorems to show the undecidability of $\textsc{Sat}(\mathsf{IF}^2)$ and, resp., $\textsc{FinSat}(\mathsf{IF}^2)$.

Theorem 4.4 ([Ber66], [Har86])**.** $\textsc{Tiling}(\mathfrak{G})$ *is Π^0_1-complete.*

Theorem 4.5 ([GK72, Lemma 2])**.** $\textsc{Tiling}(\text{Torus})$ *is Σ^0_1-complete.*

To prove the undecidability of $\textsc{Sat}(\mathsf{IF}^2)$ (Theorem 4.13) we will, for every set of tiles T, define a formula ϕ_T such that $\mathfrak{A} \models \phi_T$ iff $\mathfrak{A}$ has a T-tiling. Then we will define another formula ϕ_{grid} and show that $\mathfrak{A} \models \phi_{\text{grid}}$ iff $\mathfrak{A}$ contains (an isomorphic copy of) the grid as a substructure. Therefore $\phi_T \wedge \phi_{\text{grid}}$ is satisfiable if and only if there is a T-tiling of the grid. For the undecidability of $\textsc{FinSat}(\mathsf{IF}^2)$ (Theorem 4.20) we will define a formula ϕ_{torus} which is a modification of the formula ϕ_{grid}.

Definition 4.6. Let $T = \{t^0, \ldots, t^k\}$ be a set of tiles, and for all $i \leq k$, let $\text{right}(t^i)$ (resp. $\text{top}(t^i)$) be the set

$$\{t^j \in \{0, \ldots, k\} \mid t^i_1 = t^j_3 \text{ (resp. } t^i_0 = t^j_2)\},$$

i.e., the set of tiles matching t^i to the right (resp. top).

Then we define the first-order formulas

$$\begin{aligned}
\psi_T := \forall x \forall y \Big(\Big(H(x,y) \to \bigwedge_{i \leq k} \big(P_i(x) \to \bigvee_{t^j \in \text{right}(t^i)} P_j(y) \big) \Big) \wedge \\
\Big(V(x,y) \to \bigwedge_{i \leq k} \big(P_i(x) \to \bigvee_{t^j \in \text{top}(t^i)} P_j(y) \big) \Big) \Big),
\end{aligned}$$

$$\theta_T := \forall x \bigvee_{i \leq k} \big(P_i(x) \wedge \bigwedge_{\substack{j \leq k \\ j \neq i}} \neg P_j(x) \big) \text{ and}$$

$$\phi_T := \psi_T \wedge \theta_T,$$

over the vocabulary $V, H, P_0, \ldots, P_k$. In an IF or D context, $\phi \to \psi$ is considered to be an abbreviation of $\phi^\neg \vee \psi$, where $\phi^\neg$ is the negation normal form of $\neg\phi$.

Lemma 4.7. *Let $T = \{t_0, \ldots, t_k\}$ be a set of tiles and $\mathfrak{A} = (A, V, H)$ a structure. Then $\mathfrak{A}$ is T-tilable iff there is an expansion $\mathfrak{A}^* = (A, V, H, P_0, \ldots, P_k)$ of $\mathfrak{A}$ such that $\mathfrak{A}^* \models \phi_T$.*

Lemma 4.8. *Let $T = \{t_0, \ldots, t_k\}$ be a set of tiles and $\mathfrak{B} = (B, V, H, V', H')$ a structure. There is an FO^2 sentence γ_T of the vocabulary $\{V, H, V', H', P_0, \ldots, P_k\}$ such that $\mathfrak{B}$ is T-tilable iff there is an expansion $\mathfrak{B}^* = (A, V, H, V', H', P_0, \ldots, P_k)$ of $\mathfrak{B}$ such that $\mathfrak{B}^* \models \gamma_T$.*

Notice that ϕ_T is an FO^2-sentence. Therefore T-tiling is expressible even in FO^2. The difficulty lies in expressing that a structure is (or at least contains) a grid. This is the part of the construction where FO^2 or even D^2 formulas are no longer sufficient and the full expressivity of IF^2 is needed.

Definition 4.9. A structure $\mathfrak{A} = (A, V, H)$ is called *grid-like* iff it satisfies the conjunction ϕ_{grid} of the formulas

$$\begin{array}{lcl}
\phi_{\text{functional}}(R) & := & \forall x \forall y \big(R(y,x) \to \exists y/\{x\}\, x = y\big) \\
 & & \quad \text{for } R \in \{V, H\}, \\
\phi_{\text{injective}}(R) & := & \forall x \forall y \big(R(x,y) \to \exists y/\{x\}\, x = y\big) \\
 & & \quad \text{for } R \in \{V, H\}, \\
\phi_{\text{root}} & := & \exists x \forall y \big(\neg V(y,x) \wedge \neg H(y,x)\big), \\
\phi_{\text{distinct}} & := & \forall x \forall y\, \neg\big(V(x,y) \wedge H(x,y)\big), \\
\phi_{\text{edge}}(R, R') & := & \forall x \Big(\big(\forall y\, \neg R(y,x)\big) \to \forall y \big(R'(x,y) \to \forall x\, \neg R(x,y)\big)\Big) \\
 & & \quad \text{for } (R, R') \in \{(V,H), (H,V)\}, \\
\phi_{\text{join}} & := & \forall x \forall y \Big(\big(V(x,y) \vee H(x,y)\big) \to \exists x/\{y\}\, \big(V(y,x) \vee H(y,x)\big)\Big), \\
\phi_{\text{infinite}}(R) & := & \forall x \exists y R(x,y) \text{ for } R \in \{V, H\}.
\end{array}$$

The grid-likeness of a structure can alternatively be described in the following more intuitive way.

Remark 4.10. *A structure $\mathfrak{A} = (A, V, H)$ is grid-like iff*

i) V and H are (graphs of) injective total functions, i.e., the out-degree of every element is exactly one and the in-degree at most one (ϕ_{infinite}, $\phi_{\text{functional}}$ and $\phi_{\text{injective}}$),

ii) there is an element, called the root*, that does not have any predecessors (ϕ_{root}),*

iii) for every element, its V successor is distinct from its H successor (ϕ_{distinct}),

iv) for every element x such that x does not have a V (resp. H) predecessor, the H (resp. V) successor of x also does not have a V (resp. H) predecessor (ϕ_{edge}),

v) for every element x there is an element y such that $(x,y) \in (V \circ H) \cap (H \circ V)$ or $(x,y) \in (V \circ V) \cap (H \circ H)$,

Proof. We show that a structure $\mathfrak{B} \models \phi_{\text{grid}}$ satisfies the above five properties. The only difficult case is property v). First note that ϕ_{join} is equivalent to the first-order formula

$$\forall x \exists x' \forall y \Big(\big(V(x,y) \vee H(x,y)\big) \;\rightarrow\; \big(V(y,x') \vee H(y,x')\big) \Big).$$

Since $\phi_{\text{functional}}$, ϕ_{distinct} and ϕ_{infinite} hold as well, $\mathfrak{B}$ satisfies

$$\forall x \exists x' \exists y_1 \exists y_2 \Big(y_1 \neq y_2 \wedge V(x,y_1) \wedge H(x,y_2) \wedge \big(V(y_1,x') \vee H(y_1,x')\big) \wedge \big(V(y_2,x') \vee H(y_2,x')\big) \Big).$$

Due to $\phi_{\text{injective}}$, neither $V(y_1,x') \wedge V(y_2,x')$ nor $H(y_1,x') \wedge H(y_2,x')$ can be true if $y_1 \neq y_2$. Hence, $\mathfrak{B}$ satisfies

$$\begin{aligned}\forall x \exists x' \exists y_1 \exists y_2 \Big(y_1 \neq y_2 \wedge \Big(& \big(V(x,y_1) \wedge H(x,y_2) \wedge V(y_1,x') \wedge H(y_2,x')\big) \vee \\ & \big(V(x,y_1) \wedge H(x,y_2) \wedge H(y_1,x') \wedge V(y_2,x')\big) \Big) \Big).\end{aligned}$$

From this formula the property v) is immediate (with $x := x$ and $y := x'$). □

Now we will use Remark 4.10 to show that a grid-like structure, although it need not be the grid itself, must at least contain an isomorphic copy of the grid as a substructure.

Theorem 4.11. *Let $\mathfrak{A} = (A, V, H)$ be a grid-like structure. Then $\mathfrak{A}$ contains an isomorphic copy of $\mathfrak{G}$ as a substructure.*

Proof. If $\mathfrak{B}$ is a model with two binary relations R and R', $b \in B$ and $i \in \mathbb{N}$ then the *i-b-generated substructure* of $\mathfrak{B}$ (denoted by $\mathfrak{B}^i(b)$) is defined inductively in the following way:

$$\begin{aligned}\mathfrak{B}^0(b) &= \mathfrak{B} \restriction \{b\}, \\ \mathfrak{B}^{i+1}(b) &= \mathfrak{B} \restriction \big(B^i(b) \cup \{x \in B \mid \exists y \in B^i(b) : (y,x) \in R \cup R'\}\big).\end{aligned}$$

Let $r \in A$ be a root of $\mathfrak{A}$ (which exists because $\mathfrak{A} \models \phi_{\text{root}}$). We call a point $a \in \mathfrak{A}$ a *west border point* (resp. *south border point*) if $(r,a) \in V^n$ (resp. $(r,a) \in H^n$) for some $n \in \mathbb{N}$. Due to Remark 4.10, every point in $\mathfrak{A}$ has V- and H-in-degree at most one while the west border points have H-in-degree zero and the south border points have V-in-degree zero. We call a substructure $\mathfrak{H}$ of $\mathfrak{A}$ *in-degree complete* if every point in $\mathfrak{H}$ has the same in-degrees in $\mathfrak{H}$ as it has in $\mathfrak{A}$.

We will prove by induction that there exists a family of isomorphisms $\{f_i \mid i \in \mathbb{N}\}$ such that

1. f_i is an isomorphism from $\mathfrak{G}^i((0,0))$ to $\mathfrak{A}^i(r)$,

2. $\mathfrak{A}^i(r)$ is in-degree complete and

3. $f_{i-1} \subseteq f_i$

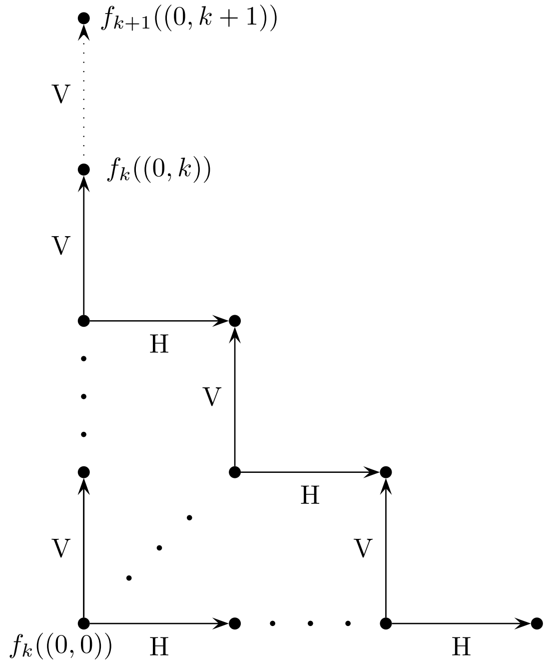

Figure 1: The inductively defined substructures

for all $i \in \mathbb{N}$.

The basis of the induction is trivial. Clearly the function f_0 defined by $f_0((0,0)) := r$ is an isomorphism from $\mathfrak{G}^0((0,0))$ to $\mathfrak{A}^0(r)$. And since r is a root it has no V- or H-predecessors. Hence, $\mathfrak{A}^0(r)$ is in-degree complete.

Let us then assume that f_k is an isomorphism from $\mathfrak{G}^k((0,0))$ to $\mathfrak{A}^k(r)$, $\mathfrak{A}^k(r)$ is in-degree complete and $f_{k-1} \subseteq f_k$. Then the k-r-generated substructure of $\mathfrak{A}^{k+1}(r)$ (which is $\mathfrak{A}^k(r)$) is isomorphic to $\mathfrak{G}^k((0,0))$ and the isomorphism is given by f_k.

We will now show how to extend f_k to the isomorphism f_{k+1}. This is done by extending f_k element by element along the diagonal (Figure 1 shows the first extension step). We will abuse notation and denote the extensions of the function f_k by h throughout the proof. We will show by induction on j that we can extend the isomorphism by assigning values for $h(j,(k+1)-j)$ for all $0 \leq j \leq k+1$ – still maintaining the isomorphism between $\mathfrak{G} \restriction \mathrm{dom}(h)$ and $\mathfrak{A} \restriction \mathrm{range}(h)$, and the in-degree completeness of $\mathfrak{A} \restriction \mathrm{range}(h)$.

Due to ϕ_{infinite} and $\phi_{\mathrm{functional}}$ the west border point $f_k((0,k))$ has a unique V-successor a. Since the k-r-generated substructure of $\mathfrak{A}^{k+1}(r)$ is isomorphic to $\mathfrak{G}^k((0,0))$ and $(0,k)$ has no V successor in $G^k((0,0))$ we know that $f_k(y) \neq a$ for every $y \in G^k((0,0))$. Note that due to ϕ_{edge} and since $f_k((0,k))$ is a west border point and has no H-predecessors in $\mathfrak{A}$, a is also a west border point and has no H-predecessor in $\mathfrak{A}$. Thus $\mathfrak{A} \restriction (\mathrm{range}(h) \cup \{a\})$ is in-degree complete. Since $\mathfrak{A} \restriction \mathrm{range}(h)$ is in-degree complete, a has no $V \cup H$-successors in $\mathrm{range}(h)$. Due to ϕ_{edge} and $\phi_{\mathrm{injective}}$, a has no reflexive loops. We extend h by $h((0,k+1)) := a$. Clearly the extended function h is an isomorphism and $\mathfrak{A} \restriction \mathrm{range}(h)$ in-degree complete.

Now let $m \in \{0, \ldots, k-1\}$ and assume that $h((j,(k+1)-j))$ is defined for all $j \leq m$, h is an isomorphism extending f_k and $h(G)$ is in-degree complete. We will prove that we can extend h by assigning a value for $h(m+1,(k+1)-(m+1))$, still maintaining the required properties. By the induction hypothesis we have defined a value for $h((m,(k+1)-m))$.

Now $h((m,(k+1)-m))$ is the V^2-successor of $h((m,(k-1)-m))$. Since $h((m,(k-1)-m))$ has no H^2 successor in the structure $\mathfrak{A} \upharpoonright \mathrm{range}(h)$, the H^2- and V^2-successors of $h((m,(k-1)-m))$ in $\mathfrak{A}$ cannot be the same point. Now by Remark 4.10v, this implies that there is a point $c \in A \setminus \mathrm{range}(h)$ such that c is the $H \circ V$- and $V \circ H$-successor of $h((m,(k-1)-m))$ in $\mathfrak{A}$. We extend h by $h((m+1,k-m)) := c$ and observe that $\mathfrak{A} \upharpoonright (\mathrm{range}(h) \cup \{c\})$ is still in-degree complete. By $\phi_{\mathrm{injective}}$ and in-degree completeness of $\mathfrak{A} \upharpoonright (\mathrm{range}(h) \setminus \{c\})$, the extended function h is an isomorphism.

Finally we extend the south border. This is possible by reasoning similar to the case where we extended the west border.

Let f_{k+1} be the isomorphism from $\mathfrak{G}^{k+1}((0,0))$ to $\mathfrak{A}^{k+1}(r)$ that exists by the inductive proof. Clearly $\mathfrak{A}^{k+1}(r)$ is in-degree complete and $f_k \subseteq f_{k+1}$. Now since the isomorphisms f_i for $i \in \mathbb{N}$ constitute an ascending chain, $\bigcup_{i \in \mathbb{N}} f_i$ is an isomorphism from $\mathfrak{G}$ to a substructure of $\mathfrak{A}$. Therefore $\mathfrak{A}$ has an isomorphic copy of the grid as a substructure. □

The last tool needed to prove the main theorem is the following trivial lemma.

Lemma 4.12. *Let T be a set of tiles and $\mathfrak{B} = (B, V, H)$ a structure. Then $\mathfrak{B}$ is T-tilable iff there is a structure $\mathfrak{A}$ which is T-tilable and contains a substructure that is isomorphic to $\mathfrak{B}$.*

The following is the main theorem of this section.

Theorem 4.13. $\textsc{Sat}(\mathsf{IF}^2)$ *is Π^0_1-complete.*

Proof. For the upper bound note that $\textsc{Sat}(\mathsf{FO}) \in \Pi^0_1$ by Gödel's completeness theorem. By Remark 2.1 it follows that $\textsc{Sat}(\mathsf{ESO}) \in \Pi^0_1$ and by the computable translation from D into ESO from [Vää07, Theorem 6.2], it follows that $\textsc{Sat}(\mathsf{D}^3) \in \Pi^0_1$. Finally, the computability of the reductions in Lemma 3.2 and Theorem 3.3 implies $\textsc{Sat}(\mathsf{IF}^2) \in \Pi^0_1$.

The lower bound follows by the reduction g from $\textsc{Tiling}(\mathfrak{G})$ to our problem defined by $g(T) := \phi_{\mathrm{grid}} \wedge \phi_T$. To see that g indeed is such a reduction, first let T be a set of tiles such that $\mathfrak{G}$ is T-tilable. Then, by Lemma 4.7, it follows that there is an expansion $\mathfrak{G}^*$ of $\mathfrak{G}$ such that $\mathfrak{G}^* \models \phi_T$. Clearly, $\mathfrak{G}^* \models \phi_{\mathrm{grid}}$ and therefore $\mathfrak{G}^* \models \phi_{\mathrm{grid}} \wedge \phi_T$. If, on the other hand, $\mathfrak{A}^*$ is a structure such that $\mathfrak{A}^* \models \phi_{\mathrm{grid}} \wedge \phi_T$, then by Theorem 4.11, the $\{V, H\}$-reduct $\mathfrak{A}$ of $\mathfrak{A}^*$ contains an isomorphic copy of $\mathfrak{G}$ as a substructure. Furthermore, by Lemma 4.7, $\mathfrak{A}$ is T-tilable. Hence, by Lemma 4.12, $\mathfrak{G}$ is T-tilable. □

4.1 Finite satisfiability for IF^2 is undecidable

We will now discuss the problem $\textsc{FinSat}(\mathsf{IF}^2)$ whose undecidability proof is similar to the above, the main difference being that it uses tilings of tori instead of tilings of the grid.

Definition 4.14. A finite structure $\mathfrak{A} = (A, V, H, V', H')$ is *torus-like* iff it satisfies the following two conditions

i) there exist unique and distinct points SW, NW, NE, SE such that
   a) SW has no V- and no H-predecessor,
   b) NW has no H-predecessor and no V-successor,
   c) NE has no V- and no H-successor and
   d) SE has no H-successor and no V-predecessor,

ii) there exist $m, n \in \mathbb{N}$ such that
   a) (A, V, H) is a model that has an isomorphic copy of the $m \times n$ grid as a component with SW, NW, NE and SE as corner points,
   b) (A, V', H) is a model that has an isomorphic copy of the $m \times 2$ grid as a component with NW, SW, SE and NE as corner points and $(NW, SW), (NE, SE) \in V'$,
   c) (A, V, H') is a model that has an isomorphic copy of the $2 \times n$ grid as a component with SE, NE, NW and SW as corner points and $(SE, SW), (NE, NW) \in H'$.

By a *component* of $\mathfrak{A} = (A, V, H)$ we mean a maximal weakly connected substructure $\mathfrak{M}$, i.e., any two points in M are connected by a path along $R := V \cup H \cup V^{-1} \cup H^{-1}$, and furthermore, for all M' such that $M \subset M' \subseteq A$, there exist two points in $\mathfrak{A} \upharpoonright M'$ that are not connected by R.

In order to define torus-likeness of a structure with an IF^2 formula we first need to express that a finite structure has a finite grid as a component. This is done in essentially the same way as expressing that a structure has a copy of the infinite grid as a substructure.

Definition 4.15. A finite structure $\mathfrak{A} = (A, V, H)$ is called *fingrid-like* iff it satisfies the conjunction ϕ_{fingrid} of the formulas

$$\begin{aligned}
\phi_{\text{SWroot}} &:= \exists x \forall y \big(\neg V(y,x) \land \neg H(y,x) \land \exists y V(x,y) \land \exists y H(x,y)\big),\\
\phi_{\text{functional}}(R) &:= \forall x \forall y \big(R(y,x) \;\rightarrow\; \exists y/\{x\}\, x = y\big)\\
&\qquad \text{for } R \in \{V, H\},\\
\phi_{\text{injective}}(R) &:= \forall x \forall y \big(R(x,y) \;\rightarrow\; \exists y/\{x\}\, x = y\big)\\
&\qquad \text{for } R \in \{V, H\},\\
\phi_{\text{distinct}} &:= \forall x \forall y\, \neg\big(V(x,y) \land H(x,y)\big),\\
\phi_{\text{SWedge}} &:= \forall x \Big(\big(\forall y\, \neg R(y,x)\big) \rightarrow \forall y \big((R'(x,y) \lor R'(y,x)) \rightarrow \forall x\, \neg R(x,y)\big)\Big)\\
&\qquad \text{for } (R, R') \in \{(V,H),(H,V)\},\\
\phi_{\text{NEedge}} &:= \forall x \Big(\big(\forall y\, \neg R(x,y)\big) \rightarrow \forall y \big((R'(x,y) \lor R'(y,x)) \rightarrow \forall x\, \neg R(y,x)\big)\Big)\\
&\qquad \text{for } (R, R') \in \{(V,H),(H,V)\},\\
\phi_{\text{finjoin}} &:= \forall x \Big(\forall y \neg V(x,y) \lor \forall y \neg H(x,y) \lor \forall y \Big(\big(V(x,y) \lor H(x,y)\big)\\
&\qquad \rightarrow \exists x/\{y\}\, \big(V(y,x) \lor H(y,x)\big)\Big)\Big),
\end{aligned}$$

Fingrid-likeness can also be described in the following intuitive way.

Remark 4.16. *A structure $\mathfrak{A} = (A, V, H)$ is fingrid-like iff*

i) *V and H are (graphs of) injective partial functions, i.e., the in- and out-degree of every element is at most one ($\phi_{\text{functional}}$ and $\phi_{\text{injective}}$),*

ii) *there exists a point, denoted by SW, that has a V-successor and an H-successor but does not have $V \cup H$-predecessors, (ϕ_{SWroot}),*

iii) *for every element, its V-successor is distinct from its H-successor (ϕ_{distinct}),*

iv) *for every element x such that x does not have a V (resp. H) predecessor, the H (resp. V) successor and predecessor of x also do not have a V (resp. H) predecessor (ϕ_{SWedge}),*

v) *for every element x such that x does not have a V (resp. H) successor, the H (resp. V) successor and predecessor of x also do not have a V (resp. H) successor (ϕ_{NEedge}),*

vi) *for every element x that has a V-successor and an H-successor there is an element y such that $(x, y) \in (V \circ H) \cap (H \circ V)$ or $(x, y) \in (V \circ V) \cap (H \circ H)$.*

Notice that for a grid $\mathfrak{G}$ to be grid-like, it is required that the grid is not of the type $1 \times n$ or $n \times 1$ for any $n \in \mathbb{N}$. A grid that is grid-like is called a *proper* grid. Now we can show that a fingrid-like structure contains a proper finite grid as a component.

Lemma 4.17. *Let $\mathfrak{A} = (A, V, H)$ be a finite fingrid-like structure. Then $\mathfrak{A}$ contains an isomorphic copy of a proper finite grid as a component.*

Proof. Due to ϕ_{SWroot} there exists a point denoted by $SW \in A$ that has a V-successor and an H-successor, but has no $V \cup H$-predecessors. Now since V is an injective partial function and A is finite, there exists $n \in \mathbb{N}$ such that for all $x \in A$ $(SW, x) \notin V^{n+1}$. For similar reasons there exists $m \in \mathbb{N}$ such that for all $x \in A$, $(SW, x) \notin H^{m+1}$. Let m and n be the smallest such numbers. We will show that $\mathfrak{A}$ has an isomorphic copy of the $m \times n$ grid as a component.

We will first show by induction on $k \leq n$ that $\mathfrak{A}$ has an isomorphic copy of the $m \times k$ grid as an in-degree complete substructure with SW as a corner point. By the selection of m the point SW has a H^i successor v_i for each $i \leq m$. Since H is an injective partial function and SW has no H-predecessors, the points v_i are all distinct and unique. Due to ϕ_{SWedge} none of the points v_i has a V-predecessor and therefore the V-successors of the points v_i are not in the set $\{v_i \mid i \leq m\}$. Therefore $\mathfrak{A} \upharpoonright \{v_i \mid i \leq m\}$ is an isomorphic copy of the $m \times 1$ grid. Due to ϕ_{SWedge}, ϕ_{SWroot} and $\phi_{\text{injective}}$ the structure $\mathfrak{A} \upharpoonright \{v_i \mid i \leq m\}$ is in-degree complete.

Let us then assume that $\mathfrak{B}$, an in-degree complete substructure of $\mathfrak{A}$, is an isomorphic copy of the $m \times k$ grid $\mathfrak{G}_{(m,k)}$ with SW as a corner point and $k < n$. Let h be the corresponding isomorphism from $\mathfrak{G}_{(m,k)}$ to $\mathfrak{B}$. We will now extend h to h' such that h'

is an isomorphism from the $m \times (k+1)$ grid to an in-degree complete substructure of $\mathfrak{A}$. Since $k+1 \leq n$ there exists a point $a_0 \in A$ such that a_0 is the V-successor of $h((0,k))$. Due to ϕ_{NEedge} and since $h((0,k))$ has a V-successor, each of the points $h((i,k))$, $i \leq m$, has a V-successor a_i. Since V is a partial injective function and the points $h((i,k))$ are all distinct, the points a_i are also all distinct. The structure $\mathfrak{B}$ is in-degree complete, and hence neither any of the points a_i nor any of their $V \cup H$-successors is in B.

We will next show that $(a_i, a_{i+1}) \in H^{\mathfrak{A}}$ for all $i < m$. For $i \leq m-2$, the point $h((i,k))$ has an H^2-successor but has no V^2-successor in the structure $\mathfrak{B}$. Therefore for all $i \leq m-2$, if the V^2-successor of $h((i,k))$ exists in $\mathfrak{A}$, it cannot be the same as the H^2-successor of $h((i,k))$. Notice that each of the points $h((i,k))$, $i \leq m-2$, has a V- and H-successor in $\mathfrak{A}$. Therefore due to ϕ_{finjoin} the $V \circ H$-successor and the $H \circ V$-successor of the point $h((i,k))$, $i \leq m-2$, are the same. Therefore the H-successor of a_i is a_{i+1} for all $i \leq m-2$.

It needs still to be shown that $(a_{m-1}, a_m) \in H^{\mathfrak{A}}$. The point $h((m-1,k))$ has no H^2 successor in $\mathfrak{A}$ since $h((m,k))$ is an east border point (due to ϕ_{NEedge} and the selection of m). Therefore there cannot be a point a in $\mathfrak{A}$ such that it is both an H^2-successor and a V^2-successor of $h((m-1,k))$. Now due to ϕ_{finjoin} and the fact that $h((m-1,k))$ has a V- and an H-successor in $\mathfrak{A}$, the $H \circ V$-successor and $V \circ H$-successor of $h((m-1,k))$ have to be the same point. Therefore $(a_{m-1}, a_m) \in H^{\mathfrak{A}}$.

We define $h' := h \cup \{((i, k+1), a_i) \mid i \leq m\}$. Each point a_i with the exception of the west border point a_0 has an H-predecessor a_{i-1}. Hence, due to the in-degree completeness of $\mathfrak{B}$, injectivity of V and H, and since each of the points a_i has a V-predecessor in the set B, we conclude that the structure $\mathfrak{A} \upharpoonright \text{range}(h')$ is an in-degree complete substructure of $\mathfrak{A}$. We also notice that due to injectivity, the points a_i have no reflexive loops. Due to in-degree completeness of B, none of the $V \cup H$-successors of the points a_i, $i \leq m$, are in the set B. Hence it is sraightforward to observe that h' is the desired isomorphism from the $m \times (k+1)$ grid to an in-degree complete substructure of $\mathfrak{A}$.

We have now proven that $\mathfrak{A}$ has an isomorphic copy of the $m \times n$ grid as a substructure with SW as a corner point. Let h be the isomorphism from the $m \times n$ grid to a substructure of $\mathfrak{A}$ with SW as a corner point. By the selection of m and n, the point $h((0,n))$ has no V-successors and $h((m,0))$ has no H-successors. Therefore, due to ϕ_{NEedge}, none of the points $h((i,n))$, $i \leq m$, have a V-successor and none of the points $h((m,j))$, $j \leq n$, have a H-successor. This together with functionality and injectivity of H and V, and the fact that west border points have no H-predecessors and south border points have no V-predecessors, imply that $\mathfrak{A}$ has an isomorphic copy of the $m \times n$ grid as a component. Since the point SW has a V-successor and an H-successor, the $m \times n$ grid is a proper grid. □

We now define some auxiliary FO^2-formulas.

$$
\begin{aligned}
\phi_{\text{NStape}} \quad &:= \quad \exists x(\phi^{(V,H)}_{\text{SW}}(x) \wedge \phi^{(V',H)}_{\text{NW}}(x)) \wedge \exists x(\phi^{(V,H)}_{\text{SE}}(x) \wedge \phi^{(V',H)}_{\text{NE}}(x)) \\
&\quad \wedge \exists x(\phi^{(V,H)}_{\text{NW}}(x) \wedge \phi^{(V',H)}_{\text{SW}}(x)) \wedge \exists x(\phi^{(V,H)}_{\text{NE}}(x) \wedge \phi^{(V',H)}_{\text{SE}}(x)) \\
&\quad \wedge \exists x \exists y(\phi^{(V,H)}_{\text{NW}}(x) \wedge \phi^{(V,H)}_{\text{SW}}(y) \wedge V'(x,y)),
\end{aligned}
$$

$$
\begin{aligned}
\phi_{\text{EWtape}} \quad &:= \quad \exists x(\phi^{(V,H)}_{\text{SW}}(x) \wedge \phi^{(V,H')}_{\text{SE}}(x)) \wedge \exists x(\phi^{(V,H)}_{\text{SE}}(x) \wedge \phi^{(V,H')}_{\text{SW}}(x)) \\
&\quad \wedge \exists x(\phi^{(V,H)}_{\text{NW}}(x) \wedge \phi^{(V,H')}_{\text{NE}}(x)) \wedge \exists x(\phi^{(V,H)}_{\text{NE}}(x) \wedge \phi^{(V,H')}_{\text{NW}}(x)) \\
&\quad \wedge \exists x \exists y(\phi^{(V,H)}_{\text{SE}}(x) \wedge \phi^{(V,H)}_{\text{SW}}(y) \wedge H'(x,y)),
\end{aligned}
$$

$$
\phi_{\text{uniquecorners}} \quad := \quad \bigwedge_{P \in C} \forall x \forall y((P(x) \wedge P(y)) \to x = y),
$$

where $C = \{\phi^{(R,S)}_{\text{T}}(x) \mid T \in \{\text{SW}, \text{NW}, \text{NE}, \text{SE}\}, (R,S) \in \{(V,H),(V',H),(V,H')\}\}$ and

$$
\begin{aligned}
\phi^{(R,S)}_{\text{SW}}(x) \quad &:= \quad \forall y\big(\neg R(y,x) \wedge \neg S(y,x)\big) \wedge \exists y R(x,y) \wedge \exists y S(x,y), \\
\phi^{(R,S)}_{\text{NW}}(x) \quad &:= \quad \forall y\big(\neg R(x,y) \wedge \neg S(y,x)\big) \wedge \exists y R(y,x) \wedge \exists y S(x,y), \\
\phi^{(R,S)}_{\text{NE}}(x) \quad &:= \quad \forall y\big(\neg R(x,y) \wedge \neg S(x,y)\big) \wedge \exists y R(y,x) \wedge \exists y S(y,x), \\
\phi^{(R,S)}_{\text{SE}}(x) \quad &:= \quad \forall y\big(\neg R(y,x) \wedge \neg S(x,y)\big) \wedge \exists y R(x,y) \wedge \exists y S(y,x),
\end{aligned}
$$

$(R,S) \in \{(V,H),(V',H),(V,H')\}$.

Let $\mathfrak{A} = (A, V, H, V', H')$ be a finite structure such that the underlying structures (A,V,H), (A,V',H) and (A,V,H') are fingrid-like. In this context the intuitive meaning of the above three formulas is the following.

- The formula $\phi_{\text{uniquecorners}}$ expresses that the structures (A,V,H), (A,V',H) and (A,V,H') each have four unique corner points, exactly one of each type, i.e., southwest corner, northwest corner, northeast corner and southeast corner. In each structure the corner points definine a boundary of a proper finite grid.

- The formula ϕ_{NStape} expresses that the proper finite grid in (A,V',H) is of the type $m \times 2$ and connects the north border of the grid in (A,V,H) to the south border of the grid in (A,V,H). (The grids in (A,V,H) and (A,V',H) form a tube.)

- The formula ϕ_{EWtape} expresses that the proper finite grid in (A,V,H') is of the type $2 \times n$ and connects the east border of the grid in (A,V,H) to the west border of the grid in (A,V,H). (The grids in (A,V,H) and (A,V,H') form a tube. The three grids together form a torus.)

Lemma 4.18. *Let $\mathfrak{A} = (A,V,H,V',H')$ be a finite structure such that the underlying structures (A,V,H), (A,V',H) and (A,V,H') are fingrid-like and the structure $\mathfrak{A}$ satisfies the conjunction of the formulas ϕ_{NStape}, ϕ_{EWtape} and $\phi_{\text{uniquecorners}}$. Then $\mathfrak{A}$ is torus-like.*

Notice that for a torus $\mathfrak{D}$ to be torus-like, it is required that the finite grid (D, V, H) is not of the type $1 \times n$ or $n \times 1$ for any $n \in \mathbb{N}$. A torus that is torus-like is called a *proper* torus.

It immediately follows from the previous lemma that there is a sentence $\phi_{\text{torus}} \in \mathsf{IF}^2$ such that for all finite structures $\mathfrak{A} = (A, V, H, V', H')$, if $\mathfrak{A} \models \phi_{\text{torus}}$ then $\mathfrak{A}$ is torus-like, and furthermore, every proper torus satisfies ϕ_{torus}.

We say that a structure $\mathfrak{A} = (A, \{R_i^{\mathfrak{A}}\}_{i \leq n})$ is a *topping* of a structure $\mathfrak{B} = (B, \{R_i^{\mathfrak{B}}\}_{i \leq n})$ iff $A = B$ and $R_i^{\mathfrak{B}} \subseteq R_i^{\mathfrak{A}}$ for all $i \leq n$.

Lemma 4.19. *Let $\mathfrak{A} = (A, V, H, V', H')$ be a finite structure with $\mathfrak{A} \models \phi_{\text{torus}}$. Then there is a torus $\mathfrak{D}$ such that $\mathfrak{A}$ contains an isomorphic copy of a topping of $\mathfrak{D}$ as a substructure.*

Proof. Immediate from Definition 4.14 and the definition of a torus, i.e., Definition 4.1. □

The following theorem is the finite analogue of Theorem 4.13.

Theorem 4.20. $\textsc{FinSat}(\mathsf{IF}^2)$ *is Σ_1^0-complete.*

Proof. For the upper bound, note that since all finite structures can be recursively enumerated and the model checking problem of IF^2 over finite models is clearly decidable, we have $\textsc{FinSat}(\mathsf{IF}^2) \in \Sigma_1^0$.

The lower bound follows by a reduction g from $\textsc{Tiling}(\text{Torus})$ to our problem defined by $g(T) := \phi_{\text{torus}} \wedge \gamma_T$. To see that g indeed is such a reduction, first let T be a set of tiles such that there is a torus $\mathfrak{D}'$ which is T-tilable. Therefore there clearly exists a proper torus $\mathfrak{D}$ that is T-tilable. Then, by Lemma 4.8, it follows that there is an expansion $\mathfrak{D}^*$ of $\mathfrak{D}$ such that $\mathfrak{D}^* \models \gamma_T$. We have $\mathfrak{D}^* \models \phi_{\text{torus}}$ and therefore $\mathfrak{D}^* \models \phi_{\text{torus}} \wedge \gamma_T$. If, on the other hand, $\mathfrak{A}^*$ is a finite structure such that $\mathfrak{A}^* \models \phi_{\text{torus}} \wedge \gamma_T$, then by Lemma 4.19, $\mathfrak{A}^*$ has a substructure $\mathfrak{B}_+^*$, which is an expansion of an isomorphic copy of a topping of a torus $\mathfrak{B}$. Furthermore, by Lemma 4.8, the $\{V, V', H, H'\}$-reduct $\mathfrak{A}$ of the structure $\mathfrak{A}^*$ is T-tilable. Hence, by the obvious analogue of Lemma 4.12, the $\{V, V', H, H'\}$-reduct $\mathfrak{B}_+$ of $\mathfrak{B}_+^*$ is T-tilable. Therefore $\mathfrak{B}$ is clearly T-tilable. □

5 Satisfiability for D^2 is NEXPTIME-complete

In this section we show that $\textsc{Sat}(\mathsf{D}^2)$ and $\textsc{FinSat}(\mathsf{D}^2)$ are *NEXPTIME*-complete. Our proof uses the fact that $\textsc{Sat}(\mathsf{FOC}^2)$ and $\textsc{FinSat}(\mathsf{FOC}^2)$ are *NEXPTIME*-complete [PH05].

Theorem 5.1. *Let τ be a relational vocabulary. For every formula $\phi \in \mathsf{D}^2[\tau]$ there is a sentence $\phi^* \in \mathsf{ESO}[\tau \cup \{R\}]$ (with* $\text{arity}(R) = |\text{Fr}(\phi)|$*),*

$$\phi^* := \exists R_1 \ldots \exists R_k \psi,$$

where R_i is of arity at most 2 and $\psi \in \mathsf{FOC}^2$, such that for all $\mathfrak{A}$ and teams X with $\text{dom}(X) = \text{Fr}(\phi)$ *it holds that*

$$\mathfrak{A} \models_X \phi \textit{ iff } (\mathfrak{A}, \text{rel}(X)) \models \phi^*, \tag{14}$$

where $(\mathfrak{A}, \mathrm{rel}(X))$ *is the expansion* $\mathfrak{A}'$ *of* $\mathfrak{A}$ *into vocabulary* $\tau \cup \{R\}$ *defined by* $R^{\mathfrak{A}'} := \mathrm{rel}(X)$.

Proof. Using induction on ϕ we will first translate ϕ into a sentence $\tau_\phi \in \mathsf{ESO}[\tau \cup \{R\}]$ satisfying (14). Then we note that τ_ϕ can be translated into an equivalent sentence ϕ^* that also satisfies the syntactic requirement of the theorem. The proof is a modification of the proof from [Vää07, Theorem 6.2]. Below we write $\phi(x, y)$ to indicate that $\mathrm{Fr}(\phi) = \{x, y\}$. Also, the quantified relations S and T below are assumed not to appear in τ_ψ and τ_θ.

1. Let $\phi(x, y) \in \{x = y, \neg x = y, P(x, y), \neg P(x, y)\}$. Then τ_ϕ is defined as
$$\forall x \forall y (R(x, y) \to \phi(x, y)).$$
2. Let $\phi(x, y)$ be of the form $=\!(x, y)$. Then τ_ϕ is defined as
$$\forall x \exists^{\leq 1} y R(x, y).$$
3. Let $\phi(x, y)$ be of the form $\neg{=}(x, y)$. Then τ_ϕ is defined as
$$\forall x \forall y \neg R(x, y).$$
4. Let $\phi(x, y)$ be of the form $\psi(x, y) \vee \theta(y)$. Then τ_ϕ is defined as
$$\exists S \exists T (\tau_\psi(R/S) \wedge \tau_\theta(R/T) \wedge \forall x \forall y (R(x, y) \to S(x, y) \vee T(y))).$$
5. Let $\phi(x)$ be of the form $\psi(x) \vee \theta$. Then τ_ϕ is defined as
$$\exists S \exists T (\tau_\psi(R/S) \wedge \tau_\theta(R/T) \wedge \forall x (R(x) \to S(x) \vee T)).$$
6. Let $\phi(x)$ be of the form $\psi(x) \wedge \theta(y)$. Then τ_ϕ is defined as
$$\exists S \exists T (\tau_\psi(R/S) \wedge \tau_\theta(R/T) \wedge \forall x \forall y (R(x, y) \to S(x) \wedge T(y))).$$
7. Let $\phi(x)$ be of the form $\exists y \psi(x, y)$. Then τ_ϕ is defined as
$$\exists S (\tau_\psi(R/S) \wedge \forall x \exists y (R(x) \to S(x, y))).$$
8. Let $\phi(x)$ be of the form $\forall y \psi(x, y)$. Then τ_ϕ is defined as
$$\exists S (\tau_\psi(R/S) \wedge \forall x \forall y (R(x) \to S(x, y))).$$

It is worth noting that in the translation above we have not displayed all the possible cases, e.g., ϕ of the form $=\!(x)$ or $P(x)$, for which τ_ϕ is defined analogously to the above. Note also that, for convenience, we allow 0-ary relations in the translation. The possible interpretations of a 0-ary relation R are $\emptyset$ and $\{\emptyset\}$. Furthermore, for a 0-ary R, we define

$\mathfrak{A} \models R$ if and only if $R^{\mathfrak{A}} = \{\emptyset\}$. Clause 5 exemplifies the use of 0-ary relations in the translation. It is easy to see that τ_ϕ in 5 is equivalent to

$$\exists S(\tau_\theta(R/\top) \vee (\tau_\psi(R/S) \wedge \forall x(R(x) \to S(x)))).$$

Furthermore, the use of 0-ary relations in the above translation can be easily eliminated with no essential change in the translation.

A straightforward induction on ϕ shows that τ_ϕ can be transformed into ϕ^* of the form

$$\exists R_1 \dots \exists R_k(\forall x \forall y \psi \wedge \bigwedge_i \forall x \exists y \theta_i \wedge \bigwedge_j \forall x \exists y^{\leq 1} R_{m_j}(x, y)),$$

where ψ and θ_i are quantifier-free. □

Note that if $\phi \in \mathsf{D}^2$ is a sentence, the relation symbol R is 0-ary and $\mathrm{rel}(X)$ (and $R^{\mathfrak{A}}$) is either $\emptyset$ or $\{\emptyset\}$. Hence, Theorem 5.1 implies that for an arbitrary sentence $\phi \in \mathsf{D}^2[\tau]$ there is a sentence $\phi^*(R/\top) \in \mathsf{ESO}[\tau]$ such that for all $\mathfrak{A}$ it holds that

$$\mathfrak{A} \models \phi \text{ iff } \mathfrak{A} \models_{\{\emptyset\}} \phi \text{ iff } \mathfrak{A} \models \phi^*(R/\top). \tag{15}$$

It is worth noting that, if $\phi \in \mathsf{D}^2$ does not contain any dependence atoms, i.e., $\phi \in \mathsf{FO}^2$, the sentence ϕ^* is of the form

$$\exists R_1 \dots \exists R_k(\forall x \forall y \psi \wedge \bigwedge_i \forall x \exists y \theta_i)$$

and the first-order part of this is in Scott normal form. So, in Theorem 5.1 we essentially translate formulas of D^2 into Scott normal form [Sco62].

Theorem 5.1 now implies the following:

Theorem 5.2. $\textsc{Sat}(\mathsf{D}^2)$ *and* $\textsc{FinSat}(\mathsf{D}^2)$ *are NEXPTIME-complete.*

Proof. Let $\phi \in \mathsf{D}^2$ be a sentence. Then, by (15), ϕ is (finitely) satisfiable if and only if ϕ^* is. Now ϕ^* is of the form

$$\exists R_1 \dots \exists R_k \psi,$$

where $\psi \in \mathsf{FOC}^2$. Clearly, ϕ^* is (finitely) satisfiable iff ψ is (finitely) satisfiable as a $\mathsf{FOC}^2[\tau \cup \{R_1, \dots, R_k\}]$ sentence. Now since the mapping $\phi \mapsto \phi^*$ is clearly computable in polynomial time and (finite) satisfiability of ψ can be checked in *NEXPTIME* [PH05], we get that $\textsc{Sat}(\mathsf{D}^2), \textsc{FinSat}(\mathsf{D}^2) \in$ *NEXPTIME*. On the other hand, since $\mathsf{FO}^2 \leq \mathsf{D}^2$ and $\textsc{Sat}(\mathsf{FO}^2)$, $\textsc{FinSat}(\mathsf{FO}^2)$ are *NEXPTIME*-hard [GKV97], it follows that $\textsc{Sat}(\mathsf{D}^2)$ and $\textsc{FinSat}(\mathsf{D}^2)$ are as well. □

6 Conclusion

We have studied the complexity of the two-variable fragments of dependence logic and independence-friendly logic. We have shown (Theorem 5.2) that both the satisfiablity

and finite satisfiability problems for D^2 are decidable, *NEXPTIME*-complete to be exact. We have also proved (Theorems 4.13 and 4.20) that both problems are undecidable for IF^2; the satisfiability and finite satisfiabity problems for IF^2 are Π_1^0-complete and Σ_1^0-complete, respectively. While the full logics D and IF are equivalent over sentences, we have shown that the finite variable variants D^2 and IF^2 are not, the latter being more expressive. This was obtained as a by-product of the deeper result concerning the decidability barrier between these two logics.

There are many open questions related to these logics. We conclude with two of them:

1. What is the complexity of the validity problems of D^2 and IF^2?

2. Is it possible to define NP-complete problems in D^2 or in IF^2?

Acknowledgments

The authors would like to thank Phokion G. Kolaitis for suggesting to study the satisfiability of D^2. The authors would also like to thank Johannes Ebbing, Lauri Hella, Allen Mann, Jouko Väänänen and Thomas Zeume for helpful discussions and comments during the preparation of this article.

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